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**Proceedings of the American Academy of Arts and Sciences.**

VOL. 54. NO. 5.— APRIL, 1919.

# A NEW GEOMETRICAL MODEL FOR THE ORTHOGONAL PROJECTION OF THE COSINES AND SINES OF COMPLEX ANGLES.

BY A. E. KENNELLY.

WITH FOUR PLATES.

*(Continued from page 3 of cover.)*

## VOLUME 54.

1. FOYE, WILBUR GARLAND.—Geological Observations in Fiji. pp. 1–145. 1 pl. September, 1918. $2.00.
2. BAILEY I. W., AND TUPPER, W. W.—Size Variation in Tracheary Cells: I. A Comparison between the secondary Xylems of vascular Cryptogams, Gymnosperms and Angiosperms. pp. 147–204. September, 1918. 75c.
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5. KENNELLY, A. E.—A new Geometrical Model for the orthogonal Projection of the Cosines and Sines of complex Angles. pp. 369–378. April 1919. 35c.

**Proceedings of the American Academy of Arts and Sciences.**

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# A NEW GEOMETRICAL MODEL FOR THE ORTHOGONAL PROJECTION OF THE COSINES AND SINES OF COMPLEX ANGLES.

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# A NEW GEOMETRICAL MODEL FOR THE ORTHOGONAL PROJECTION OF THE COSINES AND SINES OF COMPLEX ANGLES.

By A. E. Kennelly.

Received February 14, 1919. Presented February 12, 1919.

Geometrical constructions for producing plane-vector representations of cosh $(\theta_1 + i\theta_2)$, sinh $(\theta_1 + i\theta_2)$, and hence also cos $(\theta_1 + i\theta_2)$, sin $(\theta_1 + i\theta_2)$, where $(\theta_1 + i\theta_2)$ is a complex argument, or a complex "angle," have been available for some time.[1] These constructions involve a rectangular hyperbola and an associated circle, in one and the same plane, which is the plane of the drawing. By making certain projections in this plane, followed by a rotation through a quadrant, a plane-vector is produced from the origin, corresponding to the complex hyperbolic or circular sine or cosine required. This process is open to the objection that it is somewhat forced and artificial, lacking the simple projective property that a sine or cosine of a real angle possesses in either circular or hyperbolic trigonometry.

More recently, a method of deriving the hyperbolic cosine or sine of a complex angle has been obtained,[2] which has enabled the new three-dimensional model here described to be prepared. In this model, it will be seen that the cosine or sine of a complex angle, either hyperbolic or circular, can be produced, by two successive orthogonal projections on to the X Y plane, one projection being made from a rectangular hyperbola, and the other projection being then made from a particular circle definitely selected among a theoretically infinite number of such circles, all concentric at the origin O, which circles, however, are not coplanar. The selection of the particular circle is determined by the foot of the projection from the hyperbola. This effects a geometrical process which is easily apprehended and

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1 "Two Elementary Constructions in Hyperbolic Trigonometry," by A. E. Kennelly, Am. Annals of Mathematics, Salem Press, 2d Series. Vol. V, No. 4, pp. 181–184, July, 1904, mainly reproduced in "Tables of Complex Hyperbolic and Circular Functions," by A. E. Kennelly, Harvard University Press, 1914, Figs. 19–22, pp. 165–168.

2 "Artificial Electric Lines," by A. E. Kennelly, McGraw-Hill Book Co., 1917. Figs. 68 and 69, pp. 120–121.

visualized; so that once it has been realized by the student, the three-dimensional artifice is rendered superfluous, and he can roughly trace out a complex sine or cosine on an imaginary drawing board, with his eyes closed. The model, however, will be shown to enjoy certain interesting geometrical properties as a three-dimensional structure.

A photograph of the model is shown in Pl. I. On an ordinary horizontal drawing board 53.5 cm. $\times$ 31.8 cm., and 2.2 cm. thick, is a horizontal half rod A B of brass, which merely serves to support the various brass-wire semicircles, and a semihyperbola, in their proper positions. It should be understood that the axis of A B in the X Y plane, on the upper surface of the board, is a line of symmetry for the structure, which, if completed, would be formed by full circles and a complete hyperbola. For convenience, however, only half of the structure above the X Y plane is presented, the omission of the lower half being readily compensated for in the imagination of the beholder.

The eight wire semicircles are formed with the following respective radii, in decimeters: 1.0, 1.020..., 1.081..., 1.185..., 1.337..., 1.543..., 1.810..., and 2.150..., which are the respective cosines of 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, and 1.4 hyperbolic radians, according to ordinary tables of real hyperbolic functions. These successive semicircles therefore have radii equal to the cosines of successively increasing real hyperbolic angles $\theta_1$, by steps of 0.2, from 0 to 1.4 hyperbolic radians, inclusive. All of these semicircles have their common center at the origin 0, in the plane X O Y, of the drawing board. The planes of the semicircles are, however, displaced. The smallest circle of unit radius (1 decimeter), occupies the vertical plane X O Z, in which also lies the rectangular semi-hyperbola X O H. Angular distances corresponding to 0.2, 0.4,.... 1.4 hyperbolic radians, are marked off along this hyperbola at successive corresponding intervals of 0.2. The cosines of these angles, as obtainable projectively on the O X axis are marked off between C and B along the brass supporting halfbar, and at each mark, a semicircle rises from the X Y plane, at a certain angle $\beta$ with the vertical X O Z plane. This displacement angle is determined by the relation

$$\cos \beta = \frac{1}{\cosh \theta_1} = \operatorname{sech} \theta_1 \qquad \text{numeric (1)}$$

where $\theta_1$ is the particular hyperbolic angle selected. This means, as is well known, that the displacement angle $\beta$ between the plane of any semicircle and the vertical plane Z O X is equal to the gudermannian of the hyperbolic angle $\theta_1$.

The model is, of course, only a skeleton structure of 8 stages. If it could be completely developed, the number of semicircles would become infinite, and they would form a smooth continuous surface in three dimensions. Along the midplane Z O Y, all of these circles would have the same level, raised 1 decimeter above the horizontal drawing board plane of reference X O Y. The circles would increase in radius without limit, and would cover the entire X O Y plane to infinity, the hyperbola extending likewise to infinity towards its asymptote O S, in the X O Z plane. The actual model is thus the skeleton of the upper central sheet of the entire theoretical surface, near the origin.

The semicircles are also marked off in uniform steps of circular angle. Each step is taken, for convenience, as 9°, or one tenth of a quadrant. Corresponding angular steps on all of the eight semicircles are connected by thin wires, as shown in the Plates.

A front elevation of the model, taken from a point on the O Y axis $-15$ units from 0, is given in Pl. II. It will be seen that any tie wire, connecting corresponding circular angular points on the semicircles, is level, and lies at a constant height $\sin \theta_2$ decimeters above the drawing board. That is, the tie wire that connects all points of circular angle $\theta_2$, measured from O X positively towards O Y, lies at the uniform height $\sin \theta_2$ decimeters above the drawing board.

A plan view of the model, taken from a point on the O Z axis, $+15$ units above 0, is given in Pl. III. It will be seen that each and every semicircle forms an ellipse, when projected on the base plane X O Z. The semi-major axis of this ellipse has length $\cosh \theta_1$, where $\theta_1$ is the hyperbolic angle corresponding to that semicircle. The semi-minor axis is

$$\cosh \theta_1 \sin \beta = \cosh \theta_1 \cdot \tanh \theta_1 = \sinh \theta_1$$

from the well known relation that exists between a hyperbolic angle and its gudermannian circular angle; namely

$$\sin \beta = \tanh \theta_1$$

All of these ellipses have the same center of reference O. Any such system, having semi-major axes $\cosh \theta_1$, and semi-minor axes $\sinh \theta_1$, are well known to be confocal, and the foci must lie at the points $+1$ and $-1$ in the X O Y plane, or the points in which the innermost circle cuts that plane.

Moreover, as is indicated in Pl. III, the vertical shadow projections of the tiewires on the reference plane X O Y, form a system of confocal

hyperbolas, the foci occupying the already mentioned points $+1$ and $-1$, along $-\mathrm{X\,O\,X}$. Such a shadow picture is shown in Pl. IV.

*Procedure for Projecting cosh* $(\pm\theta_1 \pm i\theta_2)$

Thus premised, the process of finding the cosine of a complex hyperbolic angle $\theta_1 + i\theta_2$; that is, the process of finding cosh $(\theta_1 + i\theta_2)$ is as follows:

Find the arc C E, Pl. I, from C $= +1$ along the rectangular hyperbola C E H, which subtends $\theta_1$ radians. The hyperbolic sector comprised between the radius, O C, the hyperbolic arc, and the radius vector O E, on this arc from the origin O, will then include $\frac{\theta_1}{2}$ sq. dm. of area. Drop a vertical perpendicular from E on to O X. It will mark off a horizontal distance O D equal to cosh $\theta_1$. Proceed along the circle which rises at D, in a positive or counterclockwise direction; through $\theta_2$ circular radians, thus reaching on that circle a point G whose elevation above the drawing board is sin $\theta_2$ decimeters. The area enclosed by a radius vector from the origin O on the circle, followed between the axis O C and the circular curve, will be $\frac{\theta_2}{2}\cosh^2\theta_1$ sq. dms.

From G, drop a vertical plummet, as in Pl. II, on to the drawing board. In other words, project G orthogonally on the plane X O Y. Let $g$ be the point on the drawing board at which the plummet from G touches the surface. Then it is easily seen that O$g$ on the drawing board is the required magnitude and direction of cosh $(\theta_1 + i\theta_2)$, in decimeters, with reference to O X as the initial line in the plane X O Y. It may be read off either in rectangular coordinates along axes O X and O Y on a tracing cloth surface as shown in Pl. III, or in polar coordinates printed on a sheet seen through the tracing cloth.

If the circular angle $\theta_2$, i. e., the imaginary hyperbolic angle $i\theta_2$, lies between $\pi$ and $2\pi$ radians, (between 2 and 4 quadrants), the point G will lie on the under side of the plane X O Y, and the projection onto $g$ in that plane must be made upwards, instead of downwards.

If the hyperbolic angle whose cosine is required has a negative imaginary component, according to the expression cosh $(\theta_1 - i\theta_2)$, then starting from the projected point D, we must trace out the circular angle in the negative or clockwise direction, as viewed from the front of the model.

If the real part of the hyperbolic angle is negative, according to the expression $\cosh(-\theta_1 \pm i\theta_2)$; then since $\cosh -(\theta_1 \mp i\theta_2) = \cosh(\theta_1 \mp i\theta_2)$, we proceed as in the case of a positive real component, but with a change in the sign of the imaginary component.

The operation of tracing $\cosh(\pm\theta_1 \pm i\theta_2)$ on the X Y plane, thus calls for two successive orthogonal projections onto that plane; namely (1) the projection corresponding to $\cosh(\pm\theta_1)$ as though $i\theta_2$ did not exist, and then (2), the projection corresponding to $\cosh i\theta_2 = \cos\theta_2$ independently of $\theta_1$, except that the radius of the circle, and its plane, are both conditioned by the magnitude of $\theta_1$.

If we trace the locus of $\cosh(\theta_1 \pm i\theta_2)$, where $\theta_1$ is held constant, it is evident from an inspection of Pl. III, that we shall remain on one and the same circle, which projects into one and the same corresponding ellipse on the X Y plane. That is, the locus of $\cosh(\theta_1 \pm i\theta_2)$ with $\theta_1$ held constant, is an ellipse, whose semi major and minor diameters are $\cosh\theta_1$ and $\sinh\theta_1$ respectively. If, on the other hand, we trace $\cosh(\pm\theta_1 + i\theta_2)$ with $\theta_2$ held constant, we shall run over a certain tie wire bridging all the circles in the model, which tie wire is $\sin\theta_2$ dm. above the board, and its projection on the board, in the plane X Y of projection, is part of a hyperbola.

*Procedure for sinh* $(\theta_1 + i\theta_2)$

It would be readily possible to produce a modification of this model here described, which would enable the sine of a complex angle to be projected on the X Y plane following constructions already referred to.[3] The transition to a new model for sines is, however, unnecessary. It suffices to use the cosine model here described in a slightly different way. One has only to recall that

$$\sinh\theta = -i\cosh(\theta + i\frac{\pi}{2}) \tag{4}$$

or

$$\sinh(\theta_1 + i\theta_2) = -i\cosh\{\theta_1 + i(\theta_2 + \frac{\pi}{2})\} \tag{5}$$

Consequently, in order to find the sine of a complex hyperbolic angle, we proceed on the model as though we sought the cosine of the same angle, increased by $\frac{\pi}{2}$ radians or 1 quadrant, in the imaginary or cir-

3 Artificial Electric Lines, loc. cit. Fig. 69, page 121.

cular component. We then operate with $-i$ on the plane vector so obtained; i. e., we rotate it through 1 quadrant in the X Y plane and in the clockwise direction. An equivalent step is, however, to rotate the X and Y axes of reference in that plane through 1 quadrant in the reverse or positive direction. That is, we may omit the $-i$ operation, if, in dealing with sine projections, we treat O Y as an O X axis, and $-$O X as an O Y axis, or read off the projections on the X Y plane to the $-$Y O Y axis as initial line.

The only difference, therefore, between projecting the cosine and the sine of a complex hyperbolic angle in the model, is that in the latter case the circular component is increased by one quadrant and the projected plane vector is read off to the O Y reference axis as initial line. The model thus gives the projection of either cosh $(\pm\theta_1 \pm i\theta_2)$ or sinh $(\pm\theta_1 \pm i\theta_2)$ within the limits of $+1.4$ and $-1.4$ for $\theta_1$, and for $\theta_2$ between the limits $+\infty$ and $-\infty$. For accurate numerical work, reference would, of course, be made to the charts and Tables of such functions already published,[4] and which enable such functions to be obtained either directly or by interpolation, for all ordinary values of $\theta_1$ and $\theta_2$.

*Procedure for Projecting cos* $(\pm\theta_1 \pm i\theta_2)$.

The model enables either the cosine or sine of a hyperbolic complex angle to be projected as a complex quantity or plane vector on the X Y plane. It may also be used for projecting the cosine of a circular complex angle.

Since
$$\cos\beta = \cosh i\beta \tag{6}$$

we have, if $\beta = \pm\theta_1 \pm i\theta_2$,

$$\begin{aligned}\cos(\pm\theta_1 \pm i\theta_2) &= \cosh(\pm i\theta_1 \mp \theta_2) = \cosh(\mp\theta_2 \pm i\theta_1)\\ &= \cosh -(\pm\theta_2 \mp i\theta_1) = \cosh(\pm\theta_2 \mp i\theta_1)\end{aligned} \tag{7}$$

In projecting the circular cosine of a complex angle, therefore, we exchange the imaginary and real components, changing the sign of the latter in so doing. We then proceed as though the angle were hyperbolic. The model permits of the projection of cos $(\pm\theta_1 \pm i\theta_2)$ between the limits of $+\infty$ and $-\infty$ in $\theta_1$, and the limits of $+1.4$ and $-1.4$ in $\theta_2$.

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4 Chart Atlas of Complex Hyperbolic and Circular Functions, by A. E. Kennelly, Harvard University Press, 1914.

*Procedure for Projecting sin* $(\pm\theta_1 \pm i\theta_2)$.

Since
$$\sin\beta = \cos(\beta - \frac{\pi}{2}) \tag{8}$$

we have by substituting $\beta = \pm\theta_1 \pm i\theta_2$

$$\begin{aligned}\sin(\pm\theta_1 \pm i\theta_2) &= \cos\{(\pm\theta_1 - \frac{\pi}{2}) \pm i\theta_2\} \\ &= \cosh\{\pm\theta_2 - i(\pm\theta_1 - \frac{\pi}{2})\} \\ &= \cosh\{\pm\theta_2 + i(\frac{\pi}{2} \mp \theta_1)\} \end{aligned} \tag{9}$$

This procedure differs only from that for $\cos(\pm\theta_1 \pm i\theta_2)$ in altering the imaginary $i\theta_1$ by one quadrant.

*Relations between Complex Hyperbolic and Circular Angles.*

The projective relations above stated for the cosines and sines of both complex hyperbolic and circular angles indicate that while hyperbolic angles relate to rectangular hyperbolic sectors, and circular angles relate to circular sectors, a complex angle relates to an association of a hyperbolic and a circular sector. If the complex angle is hyperbolic, its real part relates to a hyperbolic, and its imaginary part to a circular, sector. On the other hand, if the complex angle is circular, its real part relates to a circular, and its imaginary part to a hyperbolic, sector. Complex hyperbolic trigonometry and complex circular trigonometry thus unite in a common geometrical relationship.

*Geometrical Nature of a Complex Hyperbolic Angle.*

In the engineering theory of electric conductors carrying alternating currents, complex hyperbolic angles naturally present themselves.[5] The question naturally arises as to how such complex angles may be realized and visualized geometrically. There is no difficulty in the realization of a real hyperbolic angle. The difficulty only arises with

5 The Application of Hyperbolic Functions to Electrical Engineering Problems, by A. E. Kennelly, University of London Press, 1912; also Artificial Electric Lines, McGraw-Hill Book Co., 1918.

the addition of the imaginary component. It is suggested that the new model here described permits of this realization, and thus constitutes an embodiment of a complex hyperbolic angle.

The model only purports to skeletonize part of the upper sheet of three-dimensional surface which it includes. The half beneath the X Y plane is omitted. It seems likely, moreover, that for the purpose of analytical completeness, the model should be repeated from O to infinity above the board along the $-Y$ axis. That is, it should probably have the X O Z plane as a plane of symmetry, with a rectangular hyperbola at each end of the vertical circle in that plane, and with a series of circles bending away from the vertical on each side of that plane. In the fully developed structure, each of the principal planes X O Y, X O Z, Y O Z would be a plane of symmetry. So far as projective requirements are concerned, however, this image reduplication of the model about the X Z plane is superfluous.

*Examples in the Use of the Model.*

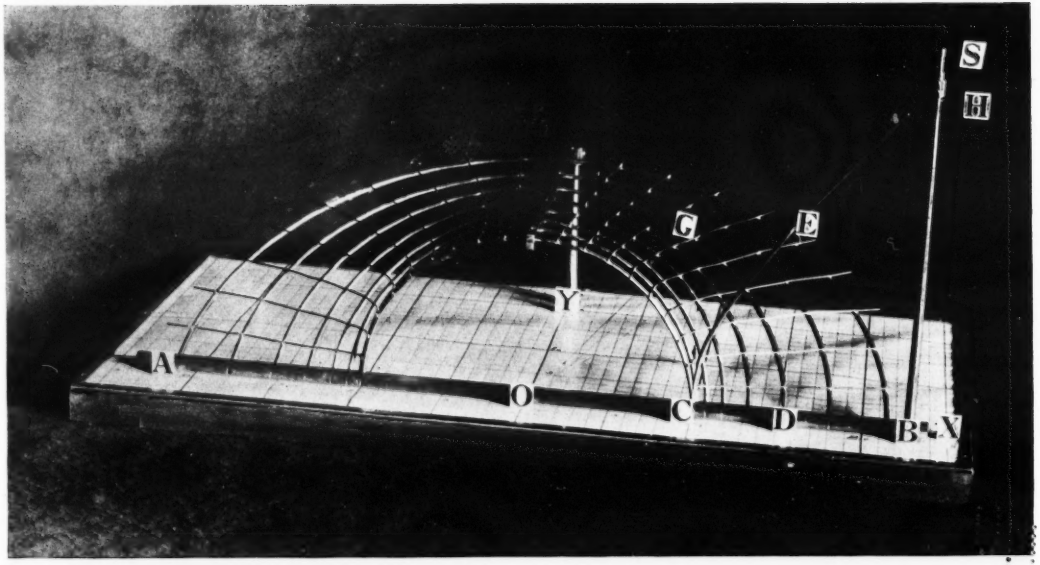
This model was designed by the writer to facilitate the conception, definition and realization of complex angles, and not for evaluating their functions; because the Tables and Charts already published are much more to the purpose of obtaining numerical values. Nevertheless, the following simple examples from the Tables may readily be checked by the reader, with reference to Pls. III and IV, but to a low degree of precision only:

$$\cosh\,(\ 1 + i\underline{0.6}) = 0.90700 + i0.95076 = 1.31400 \angle 46°.349$$
$$\sinh\,(0.8 - i\underline{0.5}) = 0.62799 - i0.94571 = 1.13522 \measuredangle 56°.414$$
$$\cos\,(\underline{0.5} + i0.5) = 0.79735 - i0.36847 = 0.87837 \measuredangle 24°.803$$
$$\sin\,(\ 0 + i1.2) = 0 \qquad + i1.50946 = 1.50946 \angle 90°$$

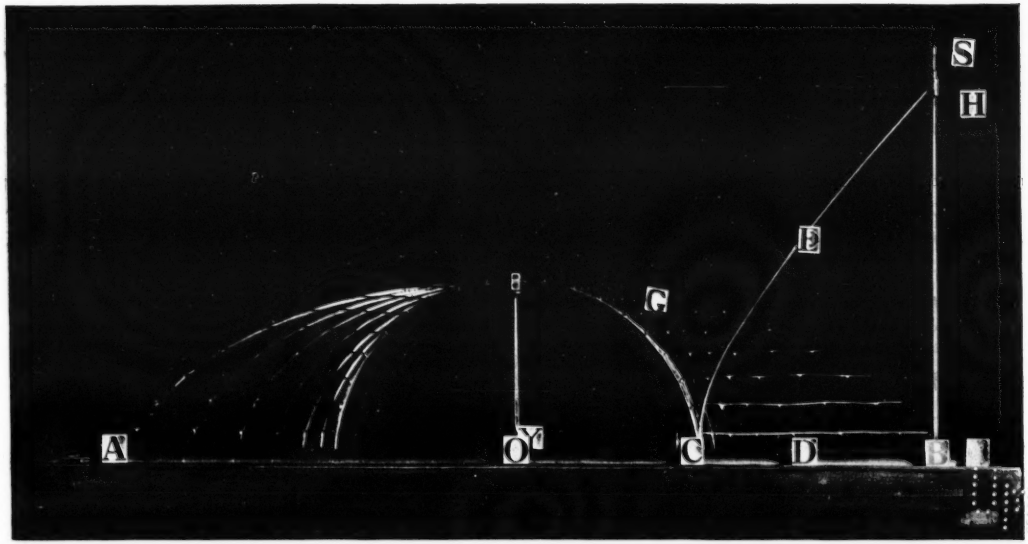
Underscored numerals represent decimal fractions of a circular quadrant. Thus $\underline{0.6} = 54°$.

*Acknowledgment.*

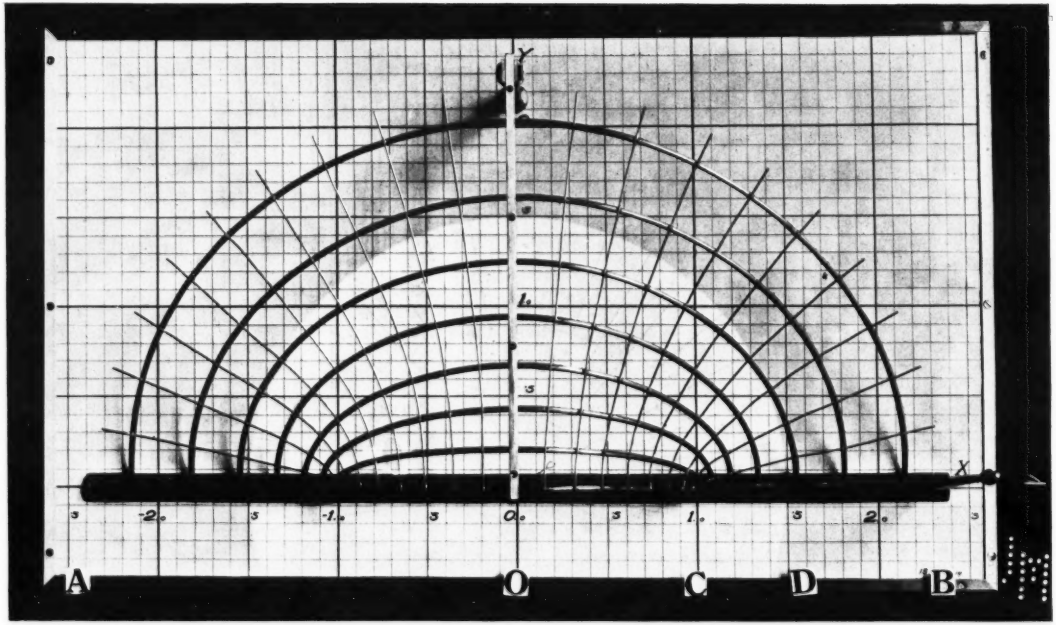
The writer is indebted to his assistant, Mr. Edy Velander, who, with care and skill, has constructed the model shown in the Plates, and also to his colleague Prof. Louis Derr, for the photographs from which the accompanying Plates have been produced.



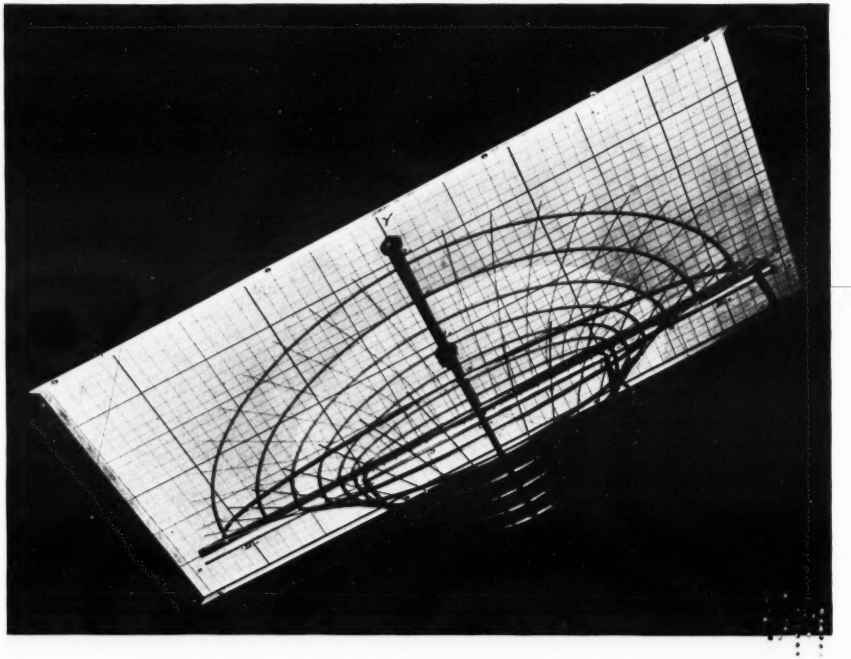

Photograph of Model.




Front Elevation of Model from a point on the OY axis —15 units (decimeters) from O.




Plan View of Model, taken from a point on the OZ axis + 15 units (decimeters) from O.




Shadow photograph of Model on the XOY plane, showing the confocal Ellipses and Hyperbolas, produced by orthogonal projection on that plane.